A Comparative Study of Some Two –Parameter Ridge-Type and Liu-Type Estimators to Combat Multicollinearity Problem in Regression Models: Simulation and Application

دراسة مقارنة لبعض مقدرات المعلمتين من النوع ريدج وليو لمواجهة مشكلة التعدد الخطى فى نماذج الانحدار: محاكاة وتطبيق

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المستخلص:

تُعد طريقة المربعات الصغرى العادية فى تحليل الانحدار الخطى المتعدد من أشهر الأساليب المستخدمة لتقدير معالم نموذج الانحدار الخطى بسبب خصائصها المفضلة، ولكنها قد تفشل عند عدم توافر فرض الاستقلالية، ويمكن إهمال هذا الفرض عند وجود ارتباط بين المتغيرات المفسرة ويُقال عند ذلك أن البيانات تتضمن مشكلة التعدد الخطى وبالتالى سوف تفقد قدرتها على الاستدلال الاحصائى، كما تصبح أساليب التقدير المتحيزة أفضل من طريقة المربعات الصغرى العادية. وقد اقترح الباحثين العديد من المقدرات للتغلب على هذه المشكلة، حيث قاموا بتطوير المقدرات المتحيزة ذات المعلمة الواحدة وكذلك ذات المعلمتين، ولكن كان لمقدرات المعلمتين مزايا أفضل من مقدرات المعلمة الواحدة حيث أن أحد المعلمتين على الأقل يكون لديه خاصية التعامل مع هذه المشكلة، ولذلك تهدف هذه الدراسة إلى اختبار أداء سبعة مقدرات حديثة وذات معلمتين لنماذج الانحدار الخطى والتى تتضمن مشكلة التعدد الخطى. وقد تم مقارنة أداء المقدرات السبعة وهم: مقدرات أولابى وآخرون (a,b 2022)، ومقدر عمارة (2022)، ومقدر أولادابو وآخرون (2022)، ومقدر مخدوم – اسلام (2023)، ومقدر أدو وآخرون (2023)، وأخيرًا

مقدر جاسم – الهيتى (2023) باستخدام مقياس متوسط مربعات الخطأ، ولذلك تم عمل محاكاة للبيانات باستخدام p= 3 , 8 ، n = 20 , 50 , 100، وتعدد خطى شبه تام عند ρ = 0.7 , 0.8 , 0.9 , 0.99 كما تم اختبار وجود الازدواج الخطى باستخدام قيم معامل تضخم التباين. وقد اتضح من النتائج التجريبية تفوق مقدر جاسم – الهيتى على بقية المقدرات تحت بعض الشروط وكذلك أفضل كفاءة لأن لديه أقل قيم لمتوسط مربعات الخطأ عند أى قيم لأحجام العينات. كما تم استخدام مجموعة من البيانات الحقيقية لتوضيح صحة النتائج الخاصة بهذه الدراسة، حيث تمت المقارنة بين النماذج المختلفة باستخدام كل من متوسط مربعات الخطأ وكذلك المتوسط النسبى للخطأ المطلق، وقد توافقت النتائج مع نتائج المحاكاة.

A Comparative Study of Some Two-Parameter Ridge-Type and Liu-Type Estimators to Combat Multicollinearity Problem in Regression Models: Simulation and Application.

Abstract

In multiple linear regression analysis, the ordinary least squares (OLS) method has been the most popular technique for estimating parameters of linear regression model due to its optimal properties. OLS estimator may fail when the assumption of independence is violated. This assumption can be violated when there is correlation between the explanatory variables. Therefore, the data is said to contain multicollinearity and eventually will mislead the inferential statistics. When multicollinearity exists, biased estimation techniques are preferable to OLS. Many authors have proposed different estimators to overcome this problem. Also, many biased estimators with one-parameter or two-parameter are developed. But, the estimators with two-parameter have advantages over that with one-parameter where they have two biasing parameters and at least one of them has the property of handling this problem impact. Therefore, this study aims to examine the performance of seven recent estimators with two-parameter of multiple linear regression model with multicollinearity problem.

The performance of the seven estimators, namely Owolabi et al. estimator (2022 a,b), Omara estimator (2022), Oladapo et al. estimator (2022), Makhdoom-Aslam estimator (2023), Idowu et al. estimator (2023), and Jassim-Alheety estimator (2023) are compared using Mean Square Error criterion. For this purpose, a simulation data with p = 3 , 8; n = 20 , 50 , 100; and full multicollinearity ρ = 0.7 , 0.8 , 0.9 , 0.99 was used. The existence of multicollinearity was evaluated usingVariance Inflation Factor (VIF) value. The empirical evidence shows that Jassim-Alheety estimator outperforms others under some conditions and is more efficient because it has the smallest MSE values in any samples sizes. A real-life dataset is used to demonstrate the findings of the paper.
The comparison was made among the different models using both the mean square error (MSE) and mean absolute percentage error (MAPE), where the results agreed with the simulation results.

Key words

Multicollinearity, Mean square error, Two-parameter estimator, Simulation, Owolabi estimators, Almost Unbiased Modified Ridge-Type Estimator (AUMRTE), Liu Dawoud-Kibria (LDK) estimator, Adaptive (K-d) class estimator (AKDE), Liu-Kibria Lukman (LKL) estimator, Modified Unbiased Optimal Estimator (MUOE).

1- Introduction

Multiple regression analysis is used to study the relationship between a single variable Y, called the response variable, and one or more explanatory variable(s). X_1 , …. , X_P by a linear model. The method of Ordinary Least Squares (OLS) estimator of model parameters is best linear unbiased estimator (BLUE) and most efficient under certain assumptions. One of the assumptions of Linear Regression model is that of independence between the explanatory variables (i.e. no multicollinearity). Violation of this assumption arises most often in regression analysis.

Multicollinearity refers to a situation in which one or more predictor variables in a multiple regression Model are highly correlated if Multicollinearity is perfect, the regression coefficients are indeterminate and their standard errors are infinite. If it is less than perfect, the regression coefficient although determinate but posses large standard errors, which means that the coefficients can not be estimated with great accuracy and appearing to have the wrong sign. Multicollinearity can be found through the concept of orthogonality, when the predictors are orthogonal or uncorrelated, all eigenvalues of the design matrix are equal to one and the design matrix is full rank. If at least one eigenvalue is different from one, especially when equal to zero or near zero, then nonorthogonality exists, meaning that multicollinearity is present. (Aslam, 2014)[1].

There are many methods used to detect multicollinearity, among these methods:

1- Compute the correlation matrix of predictors variables, a high value for the correlation between two variables may indicate that the variables are collinear. This method is easy, but it can not produce a clear estimate of the rate (degree) of multicollinearity.
2- Eigen structure of $X^{/}X$, let $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_p$ be the eigenvalues of $X^{/}X$ (in correlation form). When at least one eigenvalue is close to zero, then multicollinearity is exist.
3- Condition number: there are several methods to compute the condition number (CN) which indicate degree of multicollinearity, including of the following method:

$$CN = \left(\frac{\lambda_{max}}{\lambda_{min}} \right)^{1/2}$$

As: $\lambda_{max}, \lambda_{min}$ they represent the largest and smallest eigenvalue of $X^{/}X$, if the value of CN < 10 this means there is no problem of multicollinearity between the explanatory variables and if it is $10 < CN < 30$ then there is a problem of moderate multicollinearity between the explanatory variables and if the value $CN > 30$ this means that there is a strong multicollinearity problem between the explanatory variables (Algamal, 2021)[4].

4- Variance Inflation Factor (VIF) can be computed as follows:

$$VIF = \frac{1}{1-R_j^2}$$

Where R_j^2 is the coefficient of determination in the regression of explanatory variable X_j on the remaining explanatory variables of the model. Generally, when VIF greater than 10, we assume there exists highly multicollinearity.

Literature has suggested many alternative methods such as the Ordinary Ridge Regression (ORR) estimator (Hoerl and Kennard, 1970)[9], and the Modified Ridge Regression (MRR) estimator (Swindel, 1976)[28], etc. to address multicollinearity. The ORR estimator is one of the widely used among these estimators. It helps to overcome the problem of multicollinearity by adding a positive value (K) to the diagonal elements of the $X^/X$ matrix. This constant (K) is known as the biasing parameter or the shrinkage parameter.

Many literature is available about the selection of the biasing parameter (K). For instance, see Liu (1993)[15] proposed the biased estimator that called Liu Estimator (LE) that mingling the stein estimator and ORR estimator.

For high level of multicollinearity the matrix ($X^/X$) safer from ill condition with large condition number. The small value of ridge parameter cannot reduce the condition number by enough to overcome the ill condition. So that, Liu (2003)[16] introduced Liu-Type Estimator (LTE) that depended on two parameters make together to reduce the condition number and at the same time improve the fitting and properties of the estimator. Ozkale and Kaciranlar (2007)[25] suggest two-parameter estimator (TE), which has many features, since it contains the OLS, ORR, Liu estimators in private situations. In fact, the (ORR) and (LE) depend on OLS estimator, so researchers can use them in the case of low level of multicollinearity. Otherwise, the (LTE) and (TE) depend on any estimator. So that, researchers can use it at any level of multicollinearity. Sakallioglu and Kaciranlar (2008)[29] and Yang and Chang(2010)[31] modify the (LTE) in which it depends on (ORR). This biased estimator has superior efficient than (ORR), (LT) and (LTE). In addition, Dorugade (2014)[7] introduced the new biased estimator called ridge-type estimator (RTE). Omara (2019)[21] modify the (TE) estimator in which it depends on (ORR). Aslam and Ahmed (2020)[2] suggested the class of biased estimator modify two parameter estimator.

Lukman et al., (2019a)[17] modified the ridge-type estimator and proposed the new biased estimator called modify ridge-type estimator (MRTE). At the same time, Lukman et al. (2019b)[18] modified the ridge-type estimator with new prior information.

On the other hand, many studies go to minimize the estimators bias and at the same time keeping the MSE small. The almost unbiased estimator is one of important biased estimator which used to reduce the biased for the shrinkage estimators. In this direction, the statistical literature goes to improve the almost unbiased estimator performance by replacing the OLS estimator with more efficient shrinkage estimators. In this context Alhetty et al. (2021)[5], Algamal (2021)[4], Al-Taweel and Algamal (2022)[3] which suggests Almost Unbiased Modified Ridge-Type Estimator (AUMRTE). This estimator merges the Almost Unbiased Liu Estimator (AULE) with Modified Ridge-Type Estimator (MRTE).

Because multicollinearity is a serious problem when we need to make inferences or looking for predictive models. So it is very important for us to find a better method to deal with multicollinearity. Therefore, the **main objective** in this study, is to introduce a set of recent estimators to overcome this problem and make a comparison among them to determine which one is better to deal with this problem.

2- Methodology

In this section, the main estimation strategies will be highlighted to tackle the issue of multicollinearity.

Consider the linear regression model:

$$Y = X\beta + \varepsilon \qquad (1)$$

Where Y is an (n x 1) vector of observations on a response variable. β is a (p x 1) vector of unknown regression coefficients, X is a matrix of order (n x p) of observations on p predictor variables, and ε is (n x 1) vector of errors with E(ε) = 0 and V (ε) = $\sigma^2 I_n$. Suppose there exist an orthogonal matrix Q such that $Q'X'XQ = \Lambda =$ diag $(\lambda_1, \lambda_2, ...\lambda_p)$ where λ_i is the i^{th} eigenvalue of $X'X$. Λ and Q are the matrices of eigenvalues and eigenvectors of $X'X$, respectively.

Model (1) can be written equivalently as:

$$Y = Z\alpha + \varepsilon \qquad (2)$$

Where Z = XQ, $\alpha = Q'\beta$, and $Z'Z = \Lambda$

The Ordinary Least Square Estimator (OLSE) can be defined as:

$$\hat{\alpha} = \Lambda^{-1} X' + \varepsilon$$

2-1 Owolabi Estimator (2022a)[23]

The new estimator proposed in this study follows the works of Liu (1993)[15], and Yang and Chang (2010)[31]. While the two biasing parameters K and d have a multiplicative effect in Dorugade's (2014)[7] initial modified two-parameter estimator, they have an additive effitive in the newly proposed two-parameter estimator.

The proposed estimator is defined as follows:

$$\hat{\alpha}_{p1}(k_1, d_1) = (X^{/} X + (k+d)I)^{-1} X^{/} Y$$

$$= (\Lambda + (k+d)I)^{-1} X^{/} Y$$

$$= (\Lambda + (k+d)I)^{-1} \Lambda \hat{\alpha}_{OLS} = T_k \hat{\alpha}_{OLS} \quad (3)$$

Where $T_k = \Lambda (\Lambda + (k+d)I)^{-1}$, $K > 0$ and $0 < d < 1$

The Mean Square Error Matrix (MSEM) of the proposed estimator is defined as:

$$MSEM(\hat{\alpha}_{p1}(k_1, d_1)) = \sigma^2 T_k \Lambda^{-1} T_k^{/} + (T_k - 1)\alpha\alpha^{/} (T_k - 1)^{/} \quad (4)$$

Computation of parameters k_1 and d_1:

$$\hat{k}_1 = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d\right) \quad (5)$$

$$\hat{d}_1 = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - k\right) \quad (6)$$

The selection of the parameters d and k in $\hat{\alpha}_{p1}$ (k_1 , d_1) is obtained iteratively as follows:

Step 1: Obtain an initial estimate of d_1 using $\hat{d}^* = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right)$

Step 2: Obtain $\hat{k}_1$ from (5) using $\hat{d}^*$ in step 1

Step 3 : Estimate $\hat{d}_1$ in (6) using the $\hat{k}_1$ obtained in step 2.

Step 4 : In case $\hat{d}_1$ is not between 0 and 1, use $\hat{d}_1 = \hat{d}^*$.

The biasing parameter used in the proposed estimator P_1 (k_2 , d_2), that is k_2 and d_2 as proposed by Ozkale and Kaciranlar (2007)[25] is given by:

$$k_2 = \min\left\{ \frac{\hat{\sigma}^2}{\alpha_i^2 - d\left(\frac{\hat{\sigma}^2}{\lambda_i} + \alpha_i^2\right)} \right\} \quad (7)$$

$$d_2 = \min\left\{ \frac{\hat{\sigma}^2}{\alpha_i^2 + \frac{\hat{\sigma}^2}{\lambda_i}} \right\} \quad (8)$$

Evaluation of the performance of the proposed estimator with some existing ones OLS, Ridge estimator (1970)[9], Liu estimator (1993)[15], KL estimator (2020)[13], Dorugade estimator (2014)[7], and Two-parameter estimator by Ozkale and Kaciranlar (2007)[25] in terms of Mean Squared Error criterion was done and observed. The proposed estimator $\alpha_{p1}(k_2 , d_2)$ performs better than $\alpha_{p1}(k_1 , d_1)$ and the other six existing estimators in most cases at the different sample sizes, sigma, multicollinearity levels, and parameters.

2-2 Owolabi Estimator (2022b)[24]

The proposed two-parameter estimator of α is obtained by minimizing $(\alpha+\hat{\alpha})^{/}\ (\alpha+\hat{\alpha})$ subject to

$(Y - Z\alpha)^{/}\ (Y - Z\alpha) = C$, where C is constant.

$$(Y - Z\alpha)^{/}(Y - Z\alpha) + Kd\left[(\alpha+\hat{\alpha})^{/}(\alpha+\hat{\alpha}) - C\right] \quad (9)$$

Where K and d are langrangian multipliers.

Following Kibria and Lukman (KL)[13], the solution to (9) gives the solution to the proposed estimator as follows:

$$\hat{\alpha}_{p2}(k,d) = (\Lambda + kdI)^{-1}(\Lambda - kdI)\,\hat{\alpha}_{OLS} \quad (10)$$

$$\hat{\alpha}_{p2}(k,d) = Z_0\, Z_1\, \hat{\alpha}_{OLS} \quad (11)$$

Where $Z_0 = (\Lambda + kdI)$ and $Z_1 = (\Lambda - kdI)$, $k > 0$ and $0 < d < 1$

The MSEM of the proposed estimator is defined as:

$$MSEM\left[\hat{\alpha}_{p2}(k,d)\right] = \sigma^2 Z_0 Z_1 \Lambda^{-1} Z_0' Z_1' + (Z_0 Z_1 - I)\alpha\alpha'(Z_0 Z_1 - I)' \quad (12)$$

Computation of parameters K and d

The selection of the parameters d and k is obtained iteratively as follows:

Step 1: Obtain an initial estimate of d using $\hat{d} = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right)$

Step 2: Obtain K_{min} using $\hat{d}$ in step 1 according to Ozkale and Kaciranlar (2007)[25].

$$\hat{k}_{min} = \min\left[\frac{\hat{\sigma}^2}{d\left(2\hat{\alpha}_i^2 + \sigma^2/\lambda_i\right)}\right] \quad (13)$$

Step 3: Estimate $\hat{d}_p$ by using k_{min} in step 2.

$$\hat{d}_p = \frac{\hat{\sigma}^2}{k\left(2\hat{\alpha}_i^2 + \sigma^2/\lambda_i\right)} \quad (14)$$

Step 4: Incase $\hat{d}_p$ is not between 0 and 1 use $\hat{d}_p = \hat{d}$

In this paper, a new two-parameter estimator was proposed to solve the problem of multicollinearity for the linear regression models. The proposed estimator was compared with six other existing estimators OLS, Ridge estimator (1970), Liu estimator (1993), KL estimator (2020)[13], Modified Ridge Type estimator (2019a)[17], Two-parameter estimator by Ozkale and Kaciranlar (2007)[25]. It is obvious from the comparison that the proposed estimator performs best among the existing estimators considered in this research work using the mean square error criterion.

2-3 Omara Estimator (2022)[22]

This paper introduces an Almost Unbiased Modified Ridge-Type Estimator (AUMRTE) to avoid problems arising from multicollinearity. This estimator has an important features of the two important shrinkage estimators, the Modified Ridge-Type Estimator (MRTE) (Lukman, 2019a)[17] and Almost Unbiased Estimator (AUE) (Xu and Yang, 2011)[30].

The proposed estimator is defined as follows:

$$\hat{\alpha}_{AUMRTE}(k,d)=\left[I-k^2(1+d)^2(\Lambda+k(1+d)I)^{-2}\right]\hat{\alpha}_{OLS} \quad (15)$$

Where $\hat{\alpha}_{OLS}$ is OLS estimator.

The MSEM of the proposed estimator is formed as:

$$\text{MSEM}\,\hat{\alpha}_{AUMRTE}(k,d)=\sigma^2\sum_{i=1}^{p}\frac{1}{\lambda_i}\left(1-\frac{k^2(1+d)^2}{(\lambda_i+k(1+d))^2}\right)^2 + \sum_{i=1}^{p}\left(\frac{k^2(1+d)^2}{(\lambda_i+k(1+d))^2}\right)^2\gamma_i^2 \quad (16)$$

Choosing the shrinkage parameters (k , d):

$$d_{opt}=\frac{\lambda_i\sqrt{\frac{\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_i\hat{\gamma}_i^2}}-k\left(1-\sqrt{\frac{\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_i\hat{\gamma}_i^2}}\right)}{k\left(1-\sqrt{\frac{\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_i\hat{\gamma}_i^2}}\right)} \quad (17)$$

$$k_{opt}=\frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\sqrt{\frac{\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_i\hat{\gamma}_i^2}}}{(d+1)\left(1-\sqrt{\frac{\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_i\hat{\gamma}_i^2}}\right)} \quad (18)$$

Omara (2022)[22] used one of the important methods to obtain the optimal shrinkage parameters k and d, which is called a Generalized Cross-Validation (GCV). This method makes a equilibrium between the estimator's prediction accuracy and the bias which causes by the shrinkage of the estimator.
Theoretical comparisons were made between the AUMRTE and each of MRTE and AUE based on MSEM. These comparisons showed that the superiority of the AUMRTE over both MRTE and AUE. The simulation study results also showed that the AUMRTE is work well at the high level of correlation. For the real application, it was applied to the data of the Gross Domestic Product (GDP) of the Egyptian tourism sector. The results of the application showed that the AUMRTE improve the prediction accuracy for the model.

2-4 Oladapo Estimator (2022)[26]

The proposed biasing Liu Dawoud-Kibria (LDK) estimator for $\alpha(\hat{\alpha}_{LDK})$ is obtained by replacing the Dawoud-Kibria estimator (2020)[8] $\hat{\alpha}_{DK}$ with $\hat{\alpha}$ in the Liu estimator (1993)[15], and it becomes as follows:

$$\hat{\alpha}_{LDK} = WS\hat{\alpha} \tag{19}$$

Where W = $(\Lambda + I)^{-1}(\Lambda + dI)$,

S = $\left[\Lambda + k(1+d)I\right]^{-1}\left[\Lambda - k(1+d)I\right]$,

D and k are the biasing parameters.
The MSEM of the proposed estimator is defined as:

$$MSEM(\hat{\alpha}_{LDK}) = \sigma^2 WS\Lambda^{-1} WS + (WS - I)\alpha\alpha'(WS - I) \tag{20}$$

Determination of the Parameters k and d:

$$\hat{k}_{min} = \min\left[\frac{\hat{\sigma}^2\lambda_i(\lambda_i + d) - \hat{\alpha}_i^2\lambda_i^2(1-d)}{\hat{\sigma}^2 d(d+1) + \hat{\sigma}^2\lambda_i(d+1) + \hat{\alpha}_i^2\lambda_i(d^2+1) + 2\hat{\alpha}_i^2\lambda_i(\lambda_i d + d + \lambda_i)}\right]_{i=1}^{p} \tag{21}$$

$$d = \frac{\lambda_i(\hat{\alpha}_i^2 - \hat{\sigma}^2)}{\hat{\sigma}^2 + \lambda_i\hat{\alpha}_i^2} \tag{22}$$

Theoretical comparision of the proposed estimator with six other existing estimators OLS, Ridge estimator (1970), Liu estimator (1993), KL estimator (2020), Modified Ridge Type (MRT) estimator (2019a)[17] and Dawoud-Kibria (DK) estimator (2020)[8] shows the superiority of the proposed estimator (LDK). Results from the simulation study reveal that the proposed estimator performs better than other existing estimators used in this study, which further strengthens the theoretical study.

2-5 The Adaptive (k – d) Class Estimator (AKDCE) (Makhdoom and Aslam, 2023)[19]

Sadullah et al. (2008) suggested a biased and shrinkage estimator namely (k – d) estimator.

$$\hat{\alpha}_{kd} = (X^{/}X + KI)^{-1}(X^{/}Y + d_{op}\,\hat{\alpha}_{ORR}) \tag{23}$$

Where $k \geq 0$ and $(-\infty < d < +\infty)$. (k – d) class estimator is shrinkage estimator towards OLSE and ORRE. The optimum value to calculate d is as follows:

$$\hat{d}_{op} = \frac{\sum_{i=1}^{p} \frac{\lambda_i(\hat{\alpha}_i - \hat{\sigma}^2)}{(\lambda_i+1)^2(\lambda_i+k)}}{\sum_{i=1}^{p} \frac{\lambda_i(\lambda_i\hat{\alpha}_i^2 + \hat{\sigma}^2)}{(\lambda_i+1)^2(\lambda_i+k)^2}} \tag{24}$$

Aslam (2014)[1] used adaptive estimation procedure to fit Ridge Regression (RR) in attempt to get more efficient estimator as Adaptive Ridge Regression Estimator (ARRE).

The proposed estimator is shown below.

$$\hat{\alpha}_{ARR} = (X^{/}\hat{W}_{ARR}X + KI)^{-1}(X^{/}\hat{W}_{ARR}Y) \tag{25}$$

Where, $\hat{W}_{ARR}$ are weights assigned into diagonal matrix and called it ARR.

$$\hat{W}_{ARR} = \text{diag}\left(\frac{1}{\hat{\sigma}^2_{ARR1}}, \frac{1}{\hat{\sigma}^2_{ARR2}}, \ldots\ldots, \frac{1}{\hat{\sigma}^2_{ARRn}}\right)$$

In order to derive the Adaptive (k – d) Class Estimator (AKDCE), they extended work of Aslam et al. (2013) by replacing $\hat{\alpha}_{ORR}$ in equation (23) with $\hat{\alpha}_{ARR}$ which is given in equation (25). Resultantly, they got the Adaptive (k – d) Estimator (AKDE) as given below:

$$\hat{\alpha}_{AKDE} = (X^{/}\hat{W}_{AKD}X)^{-1}(X^{/}\hat{W}_{AKD}Y + d_{op}\,\hat{\alpha}_{ARR}) \qquad (26)$$

Where,

$$\hat{W}_{AKD} = \text{diag}\left(\frac{1}{\hat{\sigma}^2_{AKD1}}, \frac{1}{\hat{\sigma}^2_{AKD2}}, \ldots\ldots, \frac{1}{\hat{\sigma}^2_{AKDn}}\right)$$

The MSE can be numerically find by given mathematical formula in simulation:

$$MSE(\hat{\alpha}) = \sum_{i=1}^{R}\left[(\hat{\alpha}_i - \alpha)^{/}(\hat{\alpha}_i - \alpha)\right]/R \qquad (27)$$

Where R is the cumulative sum of all simulation replications.

Estimating the biasing ridge parameter

Khalaf and Shukur, 2005[14] suggested an estimator to compute biased ridge parameter k which is recognized as the "KS estimator" and is presented as:

$$\hat{K}_{KS} = \frac{\lambda_{max}\,\hat{\sigma}^2}{(n-r)\hat{\sigma}^2 + \lambda_{max}\,\hat{\alpha}^2_{OLS}\max} \qquad (28)$$

Where: λ_{max} is maximum eigen value of $X^{/}X$ matrix, $\hat{\alpha}^2_{OLS}$ max is the highest value of $\hat{\alpha}^2_{OLS}$ and $\hat{\sigma}^2$ is the MSE of residuals.

They examined the performance of the suggested estimator (AKDE) and compare it with other existing estimators OLS, Ridge estimator (1970), Adaptive Ridge Regression Estimator (ARRE) by Aslam (2014), KD estimator by Sadullah et al. (2008). It is obvious from the Monte Carlo simulation that

(AKDE) is more effective than other available estimators. As a result, when assumptions linear regression model (multicollinearity and heteroscedasticity) are being violated the AKDE class estimator is the best option over OLSE.

2-6 Idowu Estimator (2023)[10]

Liu (1993) and Kibria and Lukman (2020) proposed a Liu estimator and (K – L) estimator respectively, to improve the estimation of parameters in presence of multicollinearity. These two estimators are, however, one-parameter estimators.

The proposed Liu-Kibria Lukman (LKL) estimatory (2023)[10] following a method similar to that proposed by Yang and Change (2010), and Kaciranlar et al. (1999). The proposed estimator is obtained as follows:

$$\hat{\alpha}_{LKL} = CA\hat{\alpha} \qquad (29)$$

Where : C = $(\Lambda + I)^{-1}(\Lambda + dI)$, $A = (\Lambda + KI)^{-1}(\Lambda - KI)$,

d and K are the biasing parameters.

The MSEM of the proposed estimator is defined as:

$$MSEM(\hat{\alpha}_{LKL}) = \sigma^2 CA\Lambda^{-1}CA' + (CA - I)\alpha\alpha'(CA - I)' \qquad (30)$$

Selection of biasing parameters K and d:

For the biasing parameter k for the proposed (LKL) estimator, Idowu et al. (2023) adopted the biasing parameter k proposed by Kibria and Lukman (2020). The biasing parameter k is given as:

$$k = \min\left[\frac{\hat{\sigma}^2}{2\hat{\alpha}^2_{i,OLS} + (\sigma^2/\lambda_i)}\right] \qquad (31)$$

The optimal value of the d parameter can be considered as follows:

$$d = \frac{\lambda_i(\hat{\alpha}_i^2 - \hat{\sigma}^2) + \lambda_i k(2\hat{\alpha}_i^2\lambda_i + \hat{\alpha}_i^2 - \hat{\sigma}^2)}{\hat{\sigma}^2(\lambda_i - k) + \hat{\alpha}_i^2\lambda_i(\lambda_i - k)} \qquad (32)$$

This paper proposed a new class two-parameter estimator, namely, the Liu-Kibria Lukman (LKL) estimator, to combat multicollinearity in linear regression models. This study theoretically compares the proposed LKL estimator (2023) with some existing estimators like the OLS estimator, the ridge estimator, the Liu estimator, Kibria and Lukman (KL) estimator (2020), the Modified Ridge-Type (MRT) estimator (2019a), the two-step shrinkage (TSS) estimator (2022)[27], and the Modified New Two-Parameter (MNTP) estimator (2019)[18]. A simulation study was conducted to compare the performance of these already existing estimators with the proposed LKL estimator. From the simulation study results, the proposed LKL estimator performs better than the existing estimators.

2-7 Jassim-Alheety Estimator (2023a, b)

When there is a problem of multicollinearity or an illconditioned of design matrix in a linear regression model, many results have shown that the OLSE is no longer a good estimator, leading to the development of biased estimator such as the Ordinary Ridge Estimator (ORR) was proposed by Hoerl and Kennard (1970) as follows:

$$\begin{aligned}\hat{\alpha}_{ORR}(k) &= (X^{/}X+KI)^{-1}X^{/}Y \\ &= \left[I-K(Z+KI)^{-1}\right]\hat{\alpha}_{OLSE} \\ &= \left[I-KZ_k^{-1}\right]\hat{\alpha}_{OLSE} = W\hat{\alpha}_{OLSE}\end{aligned} \tag{33}$$

Where: $Z = X^{/}X$, $Z_k = Z + KI$,
$W = [I - K(Z + KI)^{-1}]$, $K > 0$

The Liu Estimator was proposed by Liu (1993), Almost Unbiased Ridge Etimator (AURE) was proposed by Singh and Chaubey (1986) are given by:

$$\hat{\alpha}_{Liu}(d) = (Z+I)^{-1}(Z+dI)\,\hat{\alpha}_{ORR} = F_d\ \hat{\alpha}_{OLSE} \tag{34}$$

Where $F_d = (Z+I)^{-1}(Z+dI)$

$$\begin{aligned}\hat{\alpha}_{AURE}(k) &= \left[I-k^2(Z+k)^{-2}\right]\hat{\alpha}_{OLSE} \\ &= A_k\,\hat{\alpha}_{OLSE}\end{aligned} \tag{35}$$

Where: $A_k = \left[I - K^2 (Z+K)^{-2} \right]$

Crouse et al. (1995)[6] presented the Unbiased Ridge Estimator (URE) based on the ridge estimator and prior information J, which is defined as follows:

$$\hat{\alpha}_{URE} = (Z+KI)^{-1}(X^{/}Y + KJ) \quad (36)$$

With J being uncorrelated with $\hat{\alpha}_{OLSE}$. They showed that URE estimator is unbiased estimator and its always better than OLS estimator. Jassim and Alheety (2023a)[11] proposed new three modified unbiased estimators as a generalized form depending on the last estimators and called the Modified Unbiased Ordinary Ridge Estimator (MUORE), the Modified Unbiased Ordinary Liu Estimator (MUOLE), and the Modified Unbiased Almost Unbiased Ridge Estimator (MUAURE). Then they wrote them in the following generalized form to be easy to find the statistical properties:

$$\hat{\alpha}_G = A_i\, \hat{\alpha}_{URE} \quad (37)$$

Where A_i is a positive definite matrix, I = 1 , 2 , 3 ($A_1 = W$, $A_2 = F_d$, $A_3 = A_k$).

$$MSE(\hat{\alpha}_G) = \sigma^2 A_i Z_k^{-1} A_i^{/} + (A_i - I)\,\alpha\alpha^{/}\,(A_i - I)^{/} \quad (38)$$

In addition to modifying the matrix Ai (Jassim-Alheety, 2023b)[12] introduce a modified unbiased optimal estimator (MUOE), they obtained the best choice of A_i by minimizing the MSEM of $\hat{\alpha}_G$ with respect to A_i as:

$$\tilde{A}_i = \alpha\alpha^{/}(\sigma^2 Z_k^{-1} + \alpha\alpha^{/})^{-1} \quad (39)$$

Therefore, a modified unbiased optimal estimator (MUOE) and its MSE are as follows:

$$\hat{\alpha}_{MUOE} = \tilde{A}_i\, \hat{\alpha}_{URE} \quad (40)$$

$$MSE(\hat{\alpha}_{MUOE}) = \sigma^2 \tilde{A}_i Z_k^{-1} \tilde{A}_i^{/} + (\tilde{A}_i - I)\,\alpha\alpha^{/}\,(\tilde{A}_i - I)^{/} \quad (41)$$

In Jassim-Alheety study (2023a), three biased estimators (MUORE, MUOLE and MUAURE) depending on unbiased ridge estimator (URE) in a multiple linear regression when there exists multicollinearity problem are proposed. These estimators are superior to other exists estimators (OLS, URE, ORR, Liu and AURE) which are based on sample information using the MSE Criterion. They also suggested that MUORE is the best estimator with compared to other proposed estimators.

In Jassim-Alheety study (2023b), a new unbiased rate improvement estimator (MUOE) is proposed when there is a multiple linearty problem. This estimator outperform other current estimators (MUORE, MUOLE and MUAUER) that rely on sample information. Thus the MUOE is the best estimator compared to other proposed estimators.

In Jassim-Alheety study (2023b), a new unbiased rate improvement estimator (MUOE) is proposed when there is a multiple linearty problem. This estimator outperform other current estimators (MUORE, MUOLE and MUAUER) that rely on sample information. Thus the MUOE is the best estimator compared to other proposed estimators.

3- Simulation Study

Simulation study was conducted to assess the performance of Owolabi $\left[\hat{\alpha}_{p1}(k,d), \alpha_{p2}(k,d)\right]$, Omara $(\hat{\alpha}_{AUMRTE})$, Oladapo $(\hat{\alpha}_{LDK})$, Makhdoom $(\hat{\alpha}_{AKDE})$, Idowu $(\hat{\alpha}_{LKL})$ and Jassim $(\hat{\alpha}_{MUOE})$ estimators. The explanatory variables have been generated using the following equation (Aslam, 2014).

$$X_{ij} = (1-\rho^2)^{0.5} Z_{ij} + \rho Z_{ip} \tag{42}$$

$i = 1, 2, \ldots. n$, $j = 1, 2, \ldots.., p$

where ρ represent the correlation between the explanatory variables and Z_{ij}'S are independent standard normal pseudorandom numbers. Since, we are interested in the effect of multicollinearity, in which the degrees of correlation considered more important, then four values of the pairwise correlation are considered with ρ = 0.7 , 0.8 , 0.9 and 0.99. In addition, an increase in the number of explanatory variables lead to an increase in MSE, then the number of the explanatory variables is considered as p= 3 and 8. Further, three representative values of the sample size are considered: 20, 50, and 100 because the sample size has direct impact on the prediction accuracy. The error term (ε_t) will be generated such that $\varepsilon_t \sim N(0\,,\ \sigma^2 I)$. The standard deviations in this simulation study are $\sigma = 1$, 5 , and 10. The MSE was obtained using the following equation.

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\alpha}_{ij} - \alpha_i)^{\prime}(\hat{\alpha}_{ij} - \alpha_i) \qquad (43)$$

For a combination of these different values of error variance (σ), multicollinearity levels (ρ), number of repressors (p), and sample sizes (n) the generated data is repeated R = 1000 times and the average MSE are determined.

The results obtained from the simulation study at the different specifications of σ, ρ, p, n are presented in tables 1 to 6. The best value of the averaged MSE is highlighted in bold.

Table (1): Estimated MSE values of different estimators when n = 20, p = 3

			d = 0.4							d = 0.8						
K	σ	ρ	$P_1(k,d)$	$P_2(k,d)$	AUMRTE	LDK	AKDE	LKL	MUOE	$P_1(k,d)$	$P_2(k,d)$	AUMRTE	LDK	AKDE	LKL	MUOE
0.2	1	0.7	6.745	5.828	3.726	5.693	5.392	7.672	**1.287**	6.625	5.732	3.638	5.356	5.186	6.937	**1.196**
		0.8	10.618	8.713	5.831	8.641	7.183	13.718	**2.063**	9.938	8.601	5.872	8.324	7.972	12.736	**1.893**
		0.9	12.382	10.589	5.921	10.572	8.637	14.619	**4.892**	12.362	10.482	5.922	10.379	8.438	13.973	**4.865**
		0.99	12.831	11.375	6.937	11.131	9.936	14.951	**6.308**	12.754	11.315	6.921	10.892	9.631	14.863	**6.136**
	5	0.7	7.205	6.324	3.947	5.823	5.527	8.627	**2.893**	6.975	5.863	3.927	5.502	5.318	7.238	**2.739**
		0.8	10.936	9.318	6.373	8.971	8.362	13.936	**2.962**	10.213	8.872	6.203	8.629	8.216	12.936	**2.863**
		0.9	12.917	11.372	6.682	10.916	8.815	15.728	**5.891**	12.425	10.913	6.529	10.702	8.604	14.251	**5.795**
		0.99	13.835	12.809	8.527	12.302	10.136	16.619	**8.203**	13.618	11.702	8.331	10.933	9.825	15.283	**7.917**
	10	0.7	9.182	7.365	4.251	6.931	5.922	10.625	**3.629**	8.738	6.972	4.103	6.712	5.837	9.553	**3.475**
		0.8	12.716	11.137	6.917	10.729	9.714	14.252	**4.351**	11. 139	10.816	6.725	10.305	9.134	13.972	**3.972**
		0.9	14.551	13.725	7.213	12.138	10.325	16.137	**6.793**	14.351	13.182	7.083	11.872	10.108	15.893	**6.305**
		0.99	16.827	14.633	9.715	14.625	12.174	17.734	**9.326**	15.908	14.185	9.379	13.816	11.873	17.325	**8.734**
0.6	1	0.7	6.593	5.619	3.551	5.417	3.217	7.482	**1.025**	6.325	5.426	3.316	5.018	5.015	6.518	**1.007**
		0.8	10.312	8.497	5.617	8.319	7.916	13.517	**1.826**	9.627	8.491	5.572	8.103	7.633	12.419	**1.629**
		0.9	12.197	10.283	5.729	10.392	8.519	14.316	**4.624**	12.103	10.018	5.719	10.082	8.194	13.626	**4.473**
		0.99	12.638	11.018	6.718	10.975	9.726	14.759	**6.152**	12.472	10.913	6.514	10.514	9.452	14.615	**5.839**
	5	0.7	6.958	6.082	3.663	5.617	5.272	8.429	**2.617**	6.633	5.527	3.611	5.378	4.919	6.872	**2.524**
		0.8	10.622	8.917	6.032	8.619	8.016	13.618	**2.818**	9.977	8.397	5.739	8.316	7.825	12.625	**2.619**
		0.9	12.667	11.015	6.315	10.722	8.629	15.317	**5.693**	12.013	10.616	6.338	10.352	8.436	14.013	**5.536**
		0.99	13.294	12.312	8.219	12.081	9.892	16.295	**8.014**	13.149	11.365	7.919	10.627	9.622	14.874	**7.719**
	10	0.7	8.734	6.879	3.975	6.625	5.715	10.332	**3.408**	8.372	6.629	3.825	6.319	5.613	9.372	**3.183**
		0.8	12.199	10.821	6.622	10.332	9.526	13.978	**4.076**	10.725	10.615	6.255	10.022	8.827	13.628	**3.794**
		0.9	14.315	13.335	7.019	11.873	9.971	15.729	**6.625**	13.819	12.893	6.834	11.613	9.725	15.639	**6.117**
		0.99	16.374	14.218	9.395	14.255	11.851	17.295	**9.017**	15.317	13.918	8.895	13.325	11.571	16.872	**8.339**
0.9	1	0.7	6.353	5.439	3.325	5.293	4.971	7.199	**0.974**	5.978	5.283	2.647	4.937	4.834	5.724	**0.889**
		0.8	10.182	8.378	5.516	8.017	7.638	12.827	**1.435**	9.316	8.137	4.729	7.725	6.978	11.792	**1.163**
		0.9	11.972	10.109	5.697	9.972	8.357	13.885	**4.219**	11.837	9.792	4.938	9.378	7.825	12.834	**3.875**
		0.99	12.429	10.838	6.529	10.375	9.399	14.252	**5.728**	12.328	10.358	6.152	9.972	8.744	13.837	**4.933**
	5	0.7	6.722	5.724	3.425	5.431	4.997	8.213	**2.403**	6.389	5.391	3.437	5.194	4.713	6.643	**2.371**
		0.8	10.394	8.662	5.838	8.399	7.825	13.416	**2.625**	9.625	8.102	5.516	8.192	7.698	12.416	**2.405**
		0.9	12.415	10.811	6.135	10.512	8.419	15.102	**5.397**	11.872	10.391	6.124	10.122	8.259	13.852	**5.319**
		0.99	12.971	12.014	7.918	11.871	9.614	15.917	**7.829**	12.856	11.308	7.695	10.463	9.448	14.648	**7.583**
	10	0.7	8.517	6.615	3.692	6.397	5.497	10.013	**3.213**	8.122	6.362	3.629	6.104	5.397	9.153	**2.937**
		0.8	11.899	10.662	6.451	10.182	9.312	13.722	**3.875**	10.516	10.422	5.979	9.825	8.619	13.493	**3.538**
		0.9	14.133	13.137	6.893	11.615	9.755	15.519	**6.428**	13.594	12.593	6.621	11.462	9.516	15.417	**5.932**
		0.99	16.172	14.002	9.128	13.995	11.614	16.975	**8.825**	15.022	13.617	8.597	13.133	11.274	16.659	**8.153**

Table (2): Estimated MSE values of different estimators when n = 50, p = 3

			d = 0.4							d = 0.8						
K	σ	ρ	P_1 (k , d)	P_2 (k , d)	AUMRTE	LDK	AKDE	LKL	MUOE	P_1 (k , d)	P_2 (k , d)	AUMRTE	LDK	AKDE	LKL	MUOE
0.2	1	0.7	6.416	5.517	3.473	5.372	5.001	7.337	**0.973**	6.315	5.452	3.352	5.041	4.886	6.627	**0.893**
		0.8	10.319	8.422	5.512	8.329	7.872	13.426	**1.799**	9.619	8.211	5.492	8.013	7.627	12.429	**1.537**
		0.9	12.018	10.277	5.618	10.262	8.304	14.302	**4.517**	12.002	10.172	5.712	10.028	8.106	13.621	**4.326**
		0.99	12.572	11.013	6.675	10.879	9.622	14.628	**5.988**	12.417	11.003	6.621	10.527	9.315	14.526	**5.872**
	5	0.7	6.914	6.011	3.638	5.537	5.228	8.319	**2.542**	6.638	5.526	3.589	5.229	5.003	6.913	**2.413**
		0.8	10.625	9.101	6.015	8.624	7.989	13.625	**2.627**	9.917	8.552	5.937	8.315	7.862	12.615	**2.508**
		0.9	12.623	11.021	6.376	10.672	8.501	15.401	**5.516**	12.136	10.628	6.275	10.416	8.301	13.893	**5.412**
		0.99	13.526	12.516	8.251	11.997	9.835	16.313	**7.825**	13.355	11.356	8.026	10.622	9.516	14.962	**7.639**
	10	0.7	8.842	7.015	3.972	6.625	5.618	10.315	**3.317**	8.396	6.633	3.822	6.425	5.528	9.223	**3.182**
		0.8	12.425	10.837	6.634	10.437	9.421	13.911	**4.011**	10.889	10.562	6.287	10.085	8.833	12.617	**3.657**
		0.9	14.261	13.462	6.996	11.893	9.982	15.891	**6.425**	13.919	12.856	6.725	11.527	9.801	15.526	**6.012**
		0.99	16.519	14.315	9.401	14.354	11.863	17.325	**8.989**	15.628	13.373	9.002	13.526	11.527	17.011	**8.413**
0.6	1	0.7	6.252	5.302	3.226	5.133	4.839	7.172	**0.729**	6.033	5.162	3.013	4.732	4.629	6.203	**0.712**
		0.8	10.008	8.163	5.302	8.002	7.603	13.215	**1.516**	9.375	8.176	5.272	7.863	7.303	12.135	**1.338**
		0.9	11.895	9.893	5.436	10.013	8.215	13.988	**4.317**	11.862	9.736	5.363	9.736	7.901	13.352	**4.167**
		0.99	12.317	10.822	6.492	10.655	9.416	14.336	**5.822**	12.285	10.619	6.275	10.221	9.128	14.303	**5.514**
	5	0.7	6.638	5.701	3.358	5.308	4.913	8.163	**2.304**	6.372	5.339	3.382	5.021	4.635	6.517	**2.263**
		0.8	10.317	8.629	5.716	8.311	7.792	13.302	**2.517**	9.653	8.032	5.419	8.101	7.517	12.319	**2.337**
		0.9	12.352	10.713	6.012	10.435	8.317	15.011	**5.372**	11.718	10.318	6.017	9.997	8.129	13.703	**5.216**
		0.99	12.919	12.012	7.879	11.732	9.563	15.992	**7.718**	12.773	11.029	7.625	10.351	9.318	14.536	**7.408**
	10	0.7	8.427	6.515	3.628	6.341	5.402	10.013	**3.125**	8.012	6.315	3.518	5.993	5.302	9.013	**2.937**
		0.8	11.893	10.522	6.315	10.014	9.214	13.656	**3.836**	10.417	10.328	5.917	9.729	8.517	13.355	**3.462**
		0.9	14.022	13.013	6.703	11.571	9.628	15.475	**6.319**	13.536	12.519	6.522	11.305	9.426	15.308	**5.873**
		0.99	16.035	13.893	9.027	13.913	11.527	16.918	**8.712**	15.018	13.627	8.574	13.017	11.217	16.536	**8.021**
0.9	1	0.7	5.997	5.173	3.018	4.916	4.633	6.901	**0.659**	5.627	4.902	2.316	4.622	4.527	5.416	**0.562**
		0.8	9.836	8.028	5.241	7.729	7.315	12.519	**1.128**	9.002	7.822	4.408	7.417	6.634	11.475	**0.839**
		0.9	11.638	9.801	5.372	9.635	8.012	13.574	**3.902**	11.516	9.457	4.637	9.028	7.579	12.529	**3.527**
		0.99	12.137	10.526	6.274	10.025	9.011	13.951	**5.416**	11.992	10.022	5.833	9.625	8.415	13.517	**4.629**
	5	0.7	6.415	5.417	3.137	5.139	4.627	7.912	**2.103**	6.017	5.089	3.162	4.873	4.402	6.383	**2.032**
		0.8	10.097	8.371	5.526	8.036	7.516	13.125	**2.317**	9.308	7.811	5.262	7.802	7.336	12.152	**2.138**
		0.9	12.131	10.532	5.822	10.215	8.127	14.826	**5.027**	11.526	10.021	5.839	9.785	7.903	13.578	**5.003**
		0.99	12.682	11.702	7.627	11.524	9.302	15.631	**7.519**	12.553	10.997	7.325	10.139	9.181	14.391	**7.267**
	10	0.7	8.263	6.307	3.388	6.007	5.183	9.723	**2.939**	7.795	6.002	3.318	5.824	5.027	8.803	**2.618**
		0.8	11.568	10.316	6.163	9.893	9.012	13.415	**3.536**	10.227	10.136	5.627	9.533	8.231	13.315	**3.274**
		0.9	13.827	12.882	6.557	11.324	9.425	15.233	**6.137**	13.272	12.273	6.354	11.153	9.205	15.167	**5.638**
		0.99	15.803	13.697	8.822	13.662	11.308	16.621	**8.513**	14.719	13.325	8.238	12.829	10.893	16.344	**7.925**

Table (3): Estimated MSE values of different estimators when n = 100, p = 3

			d = 0.4							d = 0.8						
K	σ	ρ	P_1 (k , d)	P_2 (k , d)	AUMRTE	LDK	AKDE	LKL	MUOE	P_1 (k , d)	P_2 (k , d)	AUMRTE	LDK	AKDE	LKL	MUOE
0.2	1	0.7	5.528	4.638	2.563	4.467	4.132	6.453	**0.659**	5.483	4.563	2.463	4.169	3.962	5.738	**0.527**
		0.8	9.463	7.317	4.637	7.452	6.951	12.568	**1.458**	8.725	7.382	4.529	7.157	6.783	11.534	**1.263**
		0.9	11.136	9.368	4.792	9.382	7.485	13.437	**3.625**	11.136	9.259	4.833	9.136	7.288	12.776	**3.469**
		0.99	11.695	10.127	5.739	9.927	8.739	13.755	**4.937**	11.527	10.118	5.749	9.628	8.436	13.698	**4.951**
	5	0.7	5.939	5.188	2.714	4.638	4.352	7.436	**1.682**	5.738	4.635	2.663	4.314	4.125	5.837	**1.538**
		0.8	9.774	8.236	5.162	7.753	6.913	12.739	**1.736**	8.839	7.629	4.917	7.452	6.943	11.728	**1.672**
		0.9	11.719	10.172	5.497	9.732	7.625	14.511	**4.626**	11.216	9.718	5.382	9.557	7.428	12.939	**4.536**
		0.99	12.663	11.635	7.338	10.879	8.614	15.428	**6.954**	12.463	10.415	7.138	9.783	8.694	13.827	**6.719**
	10	0.7	7.719	6.139	2.919	5.733	4.703	9.283	**2.463**	7.435	5.749	2.971	5.562	4.637	8.358	**2.252**
		0.8	11.521	9.931	5.728	9.529	8.522	12.968	**3.172**	9.962	9.627	5.374	9.174	7.958	11.792	**2.716**
		0.9	13.398	12.525	5.903	10.972	8.937	14.915	**5.509**	12.714	11.939	5.839	10.637	8.962	14.646	**5.138**
		0.99	15.627	13.426	8.517	13.458	10.916	16.438	**7.931**	14.729	12.463	8.136	12.657	10.623	16.132	**7.524**
0.6	1	0.7	5.388	4.429	2.319	4.257	3.955	6.219	**0.469**	5.128	4.274	2.185	3.874	3.757	5.388	**0.436**
		0.8	9.136	7.273	4.435	7.128	6.738	12.355	**1.217**	8.435	7.286	4.362	6.955	6.411	11.264	**1.021**
		0.9	10.937	8.914	4.597	9.135	7.359	12.916	**3.462**	10.933	8.833	4.495	8.858	6.973	12.449	**3.283**
		0.99	11.425	9.938	5.582	9.734	8.527	13.485	**4.937**	11.374	9.759	5.383	9.382	8.265	13.417	**4.639**
	5	0.7	5.761	4.813	2.437	4.419	4.425	7.235	**1.427**	5.417	4.493	2.472	4.114	3.727	5.639	**1.362**
		0.8	9.429	7.757	4.822	7.467	7.436	12.471	**1.635**	8.766	7.178	4.565	7.236	6.636	11.494	**1.479**
		0.9	11.474	9.892	5.151	9.535	9.517	14.165	**4.468**	10.851	9.452	5.138	8.953	7.254	12.859	**4.322**
		0.99	11.938	11.149	6.963	10.827	10.829	14.983	**6.829**	11.872	10.127	6.719	9.439	8.472	13.638	**6.571**
	10	0.7	7.572	5.672	2.784	7.439	7.464	9.177	**2.263**	7.185	5.453	2.616	4.874	4.475	8.172	**1.938**
		0.8	10.985	9.651	5.439	9.197	9.153	12.769	**2.952**	9.514	9.482	4.873	8.839	7.636	12.438	**2.527**
		0.9	13.137	12.169	5.862	10.683	10.628	14.382	**5.474**	12.639	11.661	5.696	10.462	8.529	14.453	**4.914**
		0.99	15.143	12.974	8.117	12.985	12.971	15.815	**7.891**	14.122	12.738	7.625	12.155	10.393	15.621	**7.173**
0.9	1	0.7	4.984	4.239	2.138	3.853	3.899	5.957	**0.382**	4.021	3.837	1.457	3.779	3.659	4.527	**0.295**
		0.8	8.913	7.144	4.355	8.844	6.837	11.636	**0.795**	8.137	6.966	3.528	6.528	5.736	10.563	**0.574**
		0.9	10.725	8.931	4.463	8.739	8.728	12.628	**2.938**	10.693	8.584	3.716	8.174	6.621	11.675	**2.641**
		0.99	11.273	9.608	5.388	9.194	9.156	12.971	**4.572**	10.865	9.142	4.929	8.743	7.583	12.696	**3.738**
	5	0.7	5.521	4.528	2.271	4.253	4.239	6.819	**1.257**	5.135	4.109	2.236	3.913	3.556	5.468	**1.174**
		0.8	9.155	7.457	4.619	7.161	7.194	12.263	**1.439**	8.457	6.925	4.301	6.979	6.492	11.239	**1.293**
		0.9	11.292	9.693	4.964	9.374	9.372	13.988	**4.162**	10.629	9.173	4.914	8.828	6.923	12.621	**4.162**
		0.99	11.748	10.864	6.739	10.643	10.652	14.765	**6.687**	11.693	9.901	6.428	9.263	8.251	13.493	**5.357**
	10	0.7	7.363	5.491	2.491	5.187	5.173	8.828	**1.973**	6.853	5.135	2.417	4.904	4.149	7.936	**1.719**
		0.8	10.637	9.453	5.257	8.992	8.948	12.574	**2.648**	9.365	9.257	4.735	8.613	7.317	12.421	**2.363**
		0.9	12.914	11.927	5.619	10.483	10.426	14.316	**5.251**	12.374	11.393	5.421	10.256	8.383	14.208	**4.759**
		0.99	14.959	12.734	7.937	12.725	12.758	15.738	**7.639**	13.877	12.428	7.301	11.922	9.952	15.435	**6.844**

Table (4): Estimated MSE values of different estimators when n = 20, p = 8

			d = 0.4							d = 0.8						
K	σ	ρ	$P_1(k, d)$	$P_2(k, d)$	AUMRTE	LDK	AKDE	LKL	MUOE	$P_1(k, d)$	$P_2(k, d)$	AUMRTE	LDK	AKDE	LKL	MUOE
0.2	1	0.7	8.294	7.316	5.217	7.137	6.832	9.173	**2.735**	8.153	7.288	5.159	6.841	6.626	8.489	**2.636**
		0.8	12.165	10.283	7.392	10.161	9.615	15.266	**3.517**	11.417	10.179	7.311	9.862	9.471	14.238	**3.356**
		0.9	13.839	12.122	7.438	12.028	10.173	16.152	**6.324**	13.801	11.964	7.408	11.763	9.905	15.493	**6.211**
		0.99	14.392	12.859	8.452	12.653	11.429	16.438	**7.891**	14.292	12.813	8.426	12.312	11.128	16.302	**7.636**
	5	0.7	8.762	7.872	5.433	7.382	7.028	10.125	**4.359**	8.425	7.359	5.428	7.109	6.815	8.757	**4.219**
		0.8	12.497	10.826	7.891	10.464	9.853	15.462	**4.438**	11.769	10.363	7.714	10.139	9.758	14.165	**4.304**
		0.9	14.458	12.937	8.168	12.447	10.351	17.294	**7.371**	13.971	12.492	8.028	12.253	10.167	15.714	**7.219**
		0.99	15.367	14.374	10.072	13.839	11.616	18.113	**9.742**	15.183	13.215	9.816	12.424	11.308	16.758	**9.491**
	10	0.7	10.629	8.695	5.704	8.439	7.462	12.182	**5.133**	10.227	8.458	5.679	8.214	7.385	11.137	**4.963**
		0.8	14.274	12.656	8.473	12.269	11.257	15.705	**5.859**	12.616	12.365	8.255	11.821	10.689	15.462	**5.419**
		0.9	16.051	15.274	8.783	13.625	11.827	17.623	**8.261**	15.853	14.614	8.528	12.303	11.634	17.316	**7.805**
		0.99	18.363	16.195	11.251	16.169	13.653	19.273	**10.803**	17.468	15.629	10.894	15.363	13.389	17.821	**10.253**
0.6	1	0.7	7.928	7.128	5.038	6.925	6.708	8.979	**2.525**	7.835	6.952	4.831	6.544	6.509	8.131	**2.328**
		0.8	11.852	9.973	7.129	9.872	9.471	15.015	**3.373**	11.147	9.925	7.028	9.662	9.131	13.937	**3.179**
		0.9	13.631	11.763	7.293	11.844	10.093	15.852	**6.113**	13.613	11.595	7.213	11.519	9.636	15.125	**5.913**
		0.99	13.914	12.542	8.251	12.493	11.248	16.263	**7.629**	13.902	12.439	8.019	12.173	10.905	15.993	**7.328**
	5	0.7	8.465	7.538	5.129	7.148	6.797	9.913	**4.108**	8.116	7.139	5.018	6.872	6.451	8.372	**3.975**
		0.8	12.161	10.442	7.217	10.194	9.543	15.102	**4.325**	11.438	9.844	7.162	9.805	9.393	14.179	**4.123**
		0.9	13.928	12.508	7.863	12.248	10.159	16.831	**7.166**	13.502	12.139	7.694	11.974	9.936	15.501	**6.919**
		0.99	14.751	13.893	9.797	13.553	11.373	17.729	**9.587**	14.619	12.852	9.467	12.118	11.162	16.353	**9.274**
	10	0.7	10.274	8.351	5.462	8.133	7.259	11.819	**4.926**	9.827	8.192	5.379	7.861	7.152	10.874	**4.636**
		0.8	13.652	12.365	8.142	11.828	11.061	15.473	**5.575**	12.241	12.138	7.728	11.532	10.304	15.113	**5.258**
		0.9	15.861	14.859	8.593	13.359	11.474	17.252	**8.139**	15.357	14.325	8.394	13.138	11.275	17.103	**7.611**
		0.99	17.873	15.707	9.814	15.731	13.328	18.784	**10.525**	16.813	15.429	10.369	14.821	13.139	18.318	**9.868**
0.9	1	0.7	7.814	6.937	4.882	6.759	6.451	8.673	**2.191**	7.453	6.771	4.183	6.492	6.356	7.294	**1.979**
		0.8	11.639	9.819	7.092	9.525	9.169	14.379	**2.955**	10.825	9.693	6.214	9.237	8.456	13.289	**2.662**
		0.9	13.409	11.687	7.233	11.491	9.873	15.314	**5.749**	13.389	11.236	6.415	10.828	9.307	14.393	**5.316**
		0.99	13.912	12.305	8.172	11.894	10.825	15.761	**7.215**	13.874	11.819	7.632	11.406	10.255	15.372	**6.493**
	5	0.7	8.274	7.292	4.986	6.933	6.457	9.731	**3.908**	7.806	6.848	4.903	6.673	6.217	8.129	**3.853**
		0.8	11.853	10.128	7.389	9.806	9.316	14.911	**4.197**	11.131	9.609	7.028	9.661	9.173	13.982	**3.957**
		0.9	13.928	12.393	7.684	12.013	9.939	16.607	**6.831**	13.329	11.838	7.612	11.659	9.721	15.376	**6.694**
		0.99	14.416	13.551	9.436	13.352	11.108	17.472	**9.317**	14.361	12.813	9.105	11.981	10.923	16.174	**9.136**
	10	0.7	10.017	8.163	5.191	7.872	6.921	11.568	**4.759**	9.633	7.861	3.996	7.647	6.831	10.615	**4.175**
		0.8	13.356	12.128	7.959	11.663	10.803	15.261	**5.392**	12.128	11.914	7.403	10.379	10.175	14.952	**5.028**
		0.9	15.643	14.619	8.309	13.152	11.245	16.082	**7.916**	15.159	14.103	8.159	12.913	11.052	16.966	**7.469**
		0.99	17.625	15.585	10.682	15.489	13.162	18.453	**10.348**	16.517	15.152	10.132	14.602	12.715	18.137	**9.691**

Table (5): Estimated MSE values of different estimators when n = 50, p = 8

			d = 0.4							d = 0.8						
K	σ	ρ	$P_1(k,d)$	$P_2(k,d)$	AUMRTE	LDK	AKDE	LKL	MUOE	$P_1(k,d)$	$P_2(k,d)$	AUMRTE	LDK	AKDE	LKL	MUOE
0.2	1	0.7	7.791	6.894	4.718	6.638	6.309	8.687	**1.217**	7.638	6.763	4.603	6.359	6.169	7.903	**1.184**
		0.8	11.652	9.757	6.807	9.692	9.128	14.743	**3.235**	10.901	9.511	6.719	9.366	8.973	13.728	**2.836**
		0.9	13.364	11.528	6.931	11.554	9.671	15.658	**5.868**	13.325	11.478	7.074	11.302	9.415	14.952	**5.692**
		0.99	13.813	12.386	7.957	12.106	10.953	15.996	**7.279**	13.763	12.325	7.938	11.814	10.608	15.814	**6.151**
	5	0.7	8.234	7.335	4.919	6.877	6.529	9.614	**3.894**	7.959	6.857	4.825	6.549	6.336	8.273	**3.773**
		0.8	11.975	10.451	7.332	9.919	9.274	14.907	**3.948**	11.203	9.913	7.211	9.873	9.125	13.962	**3.865**
		0.9	13.926	12.374	7.671	11.928	9.813	16.739	**6.821**	13.416	11.936	7.563	11.736	9.657	15.151	**6.759**
		0.99	14.814	13.843	9.598	13.259	10.108	17.695	**9.105**	14.638	12.682	9.382	11.919	10.811	16.208	**8.924**
	10	0.7	10.168	8.399	5.204	7.941	6.933	11.628	**4.636**	9.621	7.959	5.159	7.725	6.849	10.539	**4.439**
		0.8	13.737	12.162	7.943	11.711	10.729	15.286	**5.391**	12.174	11.841	7.571	11.379	10.136	13.914	**4.971**
		0.9	15.518	14.713	8.216	13.163	11.271	17.194	**7.744**	15.203	14.163	8.063	12.861	11.196	16.849	**7.302**
		0.99	17.805	15.682	10.767	15.672	13.136	18.672	**9.238**	16.915	14.685	10.399	14.814	12.825	18.325	**9.785**
0.6	1	0.7	7.562	6.693	4.598	6.489	6.198	8.424	**2.152**	7.362	6.459	4.373	6.156	5.917	7.507	**2.173**
		0.8	11.348	9.447	6.671	9.353	8.959	14.513	**2.866**	10.683	9.519	6.557	9.127	8.633	13.439	**2.639**
		0.9	13.171	11.106	6.715	11.397	9.596	15.205	**5.679**	13.171	11.308	6.614	11.071	9.208	14.626	**5.401**
		0.99	13.608	12.135	7.752	11.932	10.727	15.653	**7.133**	13.565	11.941	7.544	11.536	10.421	15.614	**6.855**
	5	0.7	7.905	7.173	4.603	6.685	6.208	9.416	**3.613**	7.635	6.651	4.673	6.319	5.917	7.872	**3.513**
		0.8	11.625	9.975	7.182	9.672	9.173	14.637	**3.828**	10.902	9.303	6.728	9.462	8.832	13.619	**3.625**
		0.9	13.656	11.194	7.359	11.791	9.662	16.342	**6.657**	12.018	11.618	7.351	11.255	9.478	15.053	**6.574**
		0.99	14.234	13.319	9.174	13.116	10.805	17.285	**9.196**	13.156	12.322	8.983	11.717	10.683	15.848	**8.708**
	10	0.7	9.738	7.857	4.913	7.632	6.719	11.316	**4.463**	9.332	7.651	4.882	7.238	6.625	10.302	**4.293**
		0.8	13.104	11.821	7.806	11.351	10.538	14.909	**5.182**	11.714	11.739	7.214	11.029	9.864	14.619	**4.763**
		0.9	15.369	14.337	8.129	12.814	10.914	16.762	**7.604**	14.801	13.821	7.862	12.617	10.743	16.681	**7.155**
		0.99	17.385	15.128	10.391	15.206	12.863	18.237	**10.131**	16.325	14.951	9.857	14.321	12.566	17.829	**9.317**
0.9	1	0.7	7.306	6.497	4.318	6.238	5.921	8.294	**1.996**	6.903	6.297	3.638	5.953	5.814	6.726	**1.815**
		0.8	11.143	9.352	6.553	9.125	8.893	13.805	**2.427**	10.328	9.137	5.782	8.701	7.946	12.758	**2.163**
		0.9	12.984	11.167	6.691	10.971	9.325	14.853	**5.259**	12.853	10.763	5.906	10.319	8.824	13.823	**4.879**
		0.99	13.471	11.884	7.506	11.353	10.314	15.279	**6.792**	13.233	11.315	7.173	10.929	9.705	14.816	**5.939**
	5	0.7	7.793	6.752	4.422	6.409	5.997	9.218	**3.463**	7.394	6.352	4.475	6.193	5.797	7.683	**3.305**
		0.8	11.339	9.681	6.865	9.317	8.806	14.433	**3.608**	10.691	9.105	6.518	9.163	8.688	15.448	**3.427**
		0.9	13.471	11.816	7.139	11.556	9.431	16.125	**6.314**	12.855	11.396	7.151	11.142	9.263	14.806	**6.314**
		0.99	13.952	13.138	8.962	12.828	10.619	16.916	**8.893**	13.839	12.201	8.582	11.433	10.419	15.628	**8.536**
	10	0.7	9.517	7.609	4.614	7.385	6.496	11.103	**4.256**	9.194	7.304	4.672	7.127	6.378	10.171	**3.909**
		0.8	12.825	11.673	7.403	11.173	10.362	14.771	**4.841**	11.536	11.438	6.985	10.803	9.562	14.682	**4.563**
		0.9	15.138	14.161	7.811	12.614	10.755	16.563	**7.419**	14.615	13.562	7.639	12.493	10.574	16.443	**6.928**
		0.99	17.179	14.953	10.165	14.915	12.697	17.924	**9.822**	16.125	14.617	9.517	14.119	12.106	17.697	**9.259**

Table (6): Estimated MSE values of different estimators when n = 100, p = 8

			d = 0.4							d = 0.8						
K	σ	ρ	P_1 (k , d)	P_2 (k , d)	AUMRTE	LDK	AKDE	LKL	MUOE	P_1 (k , d)	P_2 (k , d)	AUMRTE	LDK	AKDE	LKL	MUOE
0.2	1	0.7	7.084	6.208	3.925	5.839	5.571	7.875	**1.259**	6.821	5.905	3.874	5.536	5.351	7.128	**0.879**
		0.8	10.825	8.935	6.173	8.825	8.329	13.928	**2.364**	10.238	8.719	5.977	8.619	8.195	12.957	**2.165**
		0.9	12.901	10.913	6.491	10.732	8.925	14.861	**5.037**	12.532	10.808	6.259	10.466	8.613	14.143	**4.839**
		0.99	13.125	11.729	7.389	11.362	10.263	14.996	**5.728**	12.924	11.545	7.139	11.096	9.814	15.047	**5.478**
	5	0.7	7.398	6.525	4.295	6.108	5.829	8.842	**3.219**	7.153	6.028	4.017	5.791	5.526	8.275	**2.963**
		0.8	11.252	9.637	6.609	9.125	8.591	13.917	**3.536**	10.433	9.157	6.372	8.803	8.392	13.161	**3.159**
		0.9	13.105	11.513	7.154	11.133	9.039	15.837	**6.174**	12.739	11.218	6.761	10.929	8.837	14.357	**5.831**
		0.99	14.213	12.995	8.903	12.269	10.253	15.925	**7.728**	13.952	11.961	8.573	11.189	10.048	15.204	**7.496**
	10	0.7	9.157	7.528	4.527	7.163	6.278	11.688	**3.992**	8.817	7.239	4.362	6.953	6.047	11.315	**3.629**
		0.8	12.902	11.386	7.218	10.921	9.931	14.363	**4.502**	11.338	11.106	6.717	10.591	9.328	13.163	**4.158**
		0.9	14.739	13.971	7.593	12.315	10.497	16.293	**6.517**	14.174	13.328	7.253	11.914	10.368	16.077	**5.962**
		0.99	16.018	14.801	9.988	14.039	12.354	16.825	**8.896**	16.138	13.909	9.514	13.193	11.052	17.561	**8.653**
0.6	1	0.7	6.837	5.836	3.857	5.608	5.378	7.639	**1.625**	6.563	5.619	3.531	5.252	5.015	8.703	**1.427**
		0.8	10.416	8.992	5.995	8.537	8.162	13.771	**2.263**	9.892	8.716	5.737	8.348	7.873	12.681	**1.938**
		0.9	12.605	10.622	6.201	10.553	8.739	14.325	**4.921**	12.372	10.624	5.816	10.259	8.339	13.818	**4.482**
		0.99	12.994	11.897	7.118	11.282	9.968	14.983	**6.624**	13.018	11.135	6.739	10.704	9.697	14.826	**5.917**
	5	0.7	7.131	6.293	3.817	5.976	5.828	8.625	**2.837**	6.817	5.824	3.829	5.536	5.162	9.025	**2.735**
		0.8	10.837	9.165	6.236	8.832	8.819	13.829	**3.025**	10.163	8.916	5.927	8.619	8.017	12.826	**2.869**
		0.9	12.852	11.216	6.583	10.974	9.914	15.573	**5.861**	12.255	10.802	6.558	10.309	8.633	14.293	**5.733**
		0.99	13.326	12.537	8.392	12.215	11.205	16.327	**8.294**	13.219	11.519	8.193	10.825	9.825	15.015	**7.925**
	10	0.7	9.993	9.021	4.153	8.927	8.123	10.528	**3.618**	8.527	6.836	4.028	6.208	5.827	9.513	**3.302**
		0.8	12.319	11.139	6.879	10.531	10.136	14.163	**4.357**	11.936	10.925	6.211	10.253	9.019	13.822	**3.961**
		0.9	14.427	13.574	7.251	12.238	12.027	15.715	**6.822**	14.018	13.172	7.036	11.859	9.936	15.837	**6.372**
		0.99	16.514	14.992	9.599	14.807	14.312	17.294	**9.139**	15.506	14.203	9.131	13.508	11.738	17.038	**8.423**
0.9	1	0.7	6.258	5.637	3.538	5.264	5.193	7.339	**1.728**	5.437	5.314	2.895	5.269	5.036	5.996	**1.608**
		0.8	10.371	8.523	5.726	8.408	8.217	13.025	**2.139**	9.588	8.322	4.919	7.925	7.154	11.928	**1.939**
		0.9	12.173	10.319	5.863	10.227	9.142	14.121	**4.353**	12.029	9.903	5.192	9.529	8.139	13.015	**4.025**
		0.99	12.729	11.025	6.712	10.937	10.593	14.434	**5.927**	12.291	10.531	6.387	10.163	8.992	14.137	**5.139**
	5	0.7	6.934	5.931	3.625	5.692	5.425	8.372	**2.639**	6.571	5.516	3.615	5.302	4.953	7.872	**2.536**
		0.8	10.516	8.822	6.129	8.571	8.334	13.671	**2.894**	9.834	8.529	5.711	8.335	7.821	12.639	**2.698**
		0.9	12.638	11.136	6.394	10.933	10.711	15.366	**5.627**	12.025	10.521	6.309	10.014	8.305	14.085	**5.536**
		0.99	13.155	12.338	8.205	12.182	11.024	16.192	**8.014**	13.149	11.372	7.824	10.663	9.631	15.926	**6.734**
	10	0.7	8.729	6.827	3.854	6.529	6.304	10.226	**3.327**	8.286	6.527	3.851	6.308	5.529	9.374	**2.908**
		0.8	12.011	10.815	6.639	10.327	9.169	13.961	**4.025**	11.713	10.613	6.109	10.133	8.773	13.821	**3.726**
		0.9	14.308	13.316	7.028	11.916	11.327	15.802	**6.636**	14.755	12.729	6.826	11.656	9.724	17.673	**6.194**
		0.99	15.373	15.166	9.397	14.251	13.169	17.139	**8.902**	15.216	13.825	8.735	13.359	11.351	18.819	**8.285**

It is shown from tables 1-6 the following:

1- For any sample size (n), standard deviation (σ), and number of predictor variables (p), the MUOE estimator gives the smallest MSE for the simulation conditions.

Therefore, the results show that MUOE estimator is performing better than the rest of the estimators, followed by AUMRTE and AKDE estimators.

2- The P_2 (k , d) estimator performance is between the LDK and P_1 (k , d) estimators, while the LKL estimator gives the highest MSE values and performs the worst among all estimators.

3- Regarding the number of explanatory variables p, one can see that there is a positive impact on MSE, where there are increasing in MSE values when the p increasing from three to eight variables.

4- With the increase in standard deviation (σ), the MSE of all the estimators increases generally for all levels of sample size, multicollinearity, and number of predictors. This is also evident from Figure (1).

5- The increase in the sample size (n) impacted MSE values of all the estimators to decrease, regardless the values of ρ , σ , and p. This is also evident from Figure (2).

6- It was observed that the MSE values of the estimators increases as the level of correlation (ρ) increases. This is also evident from Figure (3).

Also, as the biasing parameters (k , d) increases, a decrease in the MSE values was noticed.

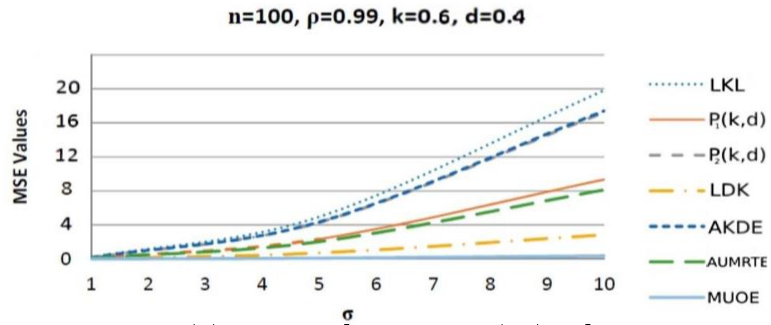

Figure (1): MSE values versus (σ) values

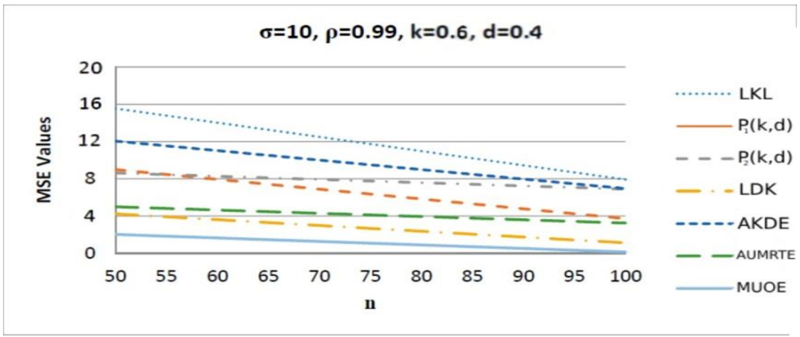

Figure (2): MSE values versus n values

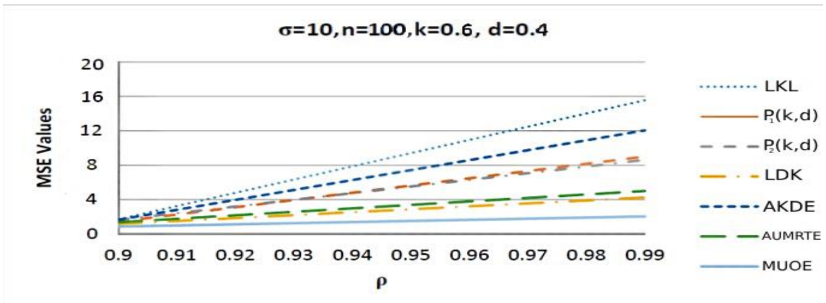

Figure (3): MSE values versus ρ values

4- Real Data Application

Given that inflation is one of the most important problems facing the Egyptian economy, determining the factors affecting it is very important. Inflation rate based on the consumer price index will represent the dependent variable (Y). The independent variables are: exchange rates (X_1), interest rates (X_2), money supply M_2 (% of GDP) (X_3), government spending (% of GDP) (X_4).

Annual data covering the period 2000-2023 were obtained using the databases of both the central bank of Egypt and the world bank.

The regression model for these data is defined as:

$$Y_i = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \varepsilon_i \quad (44)$$

The correlation matrix of the predictor variables is given in table(7).

Table (7): Correlation Matrix

	X_1	X_2	X_3	X_4
X_1	1			
X_2	0.5716	1		
X_3	0.3902	0.8325	1	
X_4	0.2514	0.7179	0.1753	1

In table (7) the correlation matrix is shown that indicates a multicollinearity problem since the pairwise correlations reach up to 0.8325 between interest rates (X_2) and money supply (X_3), this demonstrated by increasing the value of Variance Inflation Factor(VIF) for the variables X_2 and X_3.

Also, variance inflation factors (VIFs) and a condition number are adopted to diagnose multicollinearity in the model. The VIFs are : VIF (X_1) = 79.85, VIF (X_2) = 574.83, VIF (X_3) = 452.61, and VIF (X_4) = 163.72 which indicate that there is a strong multicollinearity. The matrix $X^/X$ has singular values (eigenvalues): λ_1 = 208.304, λ_2 = 961.502, λ_3 = 72.938, λ_4 = 87536.713.

The condition number defined as $\sqrt{\lambda_{max} / \lambda_{min}}$ equals 34.64. Both tests are evidence that the model possesses severe multicollinearity.

Table (8): Regression coefficients and the corresponding MSE values for OLS and used estimators

	OLS	$P_1(k, d)$	$P_2(k, d)$	AUMRTE	LDK	AKDE	LKL	MUOE
$\hat{\alpha}_0$	13.9638	2.3071	7.9652	5.6322	-3.8252	10.3274	8.9325	6.3202
$\hat{\alpha}_1$	7.9285*	0.8638	2.1638	0.8369*	-6.2715*	1.6382	2.0301*	0.7395*
$\hat{\alpha}_2$	3.0282	1.3627*	0.9631*	1.6315*	-1.7391*	0.7258*	0.9375*	0.8851*
$\hat{\alpha}_3$	0.0025*	0.6254*	0.5274*	0.3927*	-8.1792*	0.2829*	0.7251*	0.3927*
$\hat{\alpha}_4$	0.0104	0.5713	0.6327	0.0193*	-0.6358	0.2501*	0.7038	0.5033
K	-	2.9362	0.4371	3.0152	0.0037	0.1053	0.0063	0.8315
d	-	0.6018	0.0874	0.1327	0.3975	0.1859	0.729	0.6038
MSE	971.62	387.26	389.03	239.17	339.18	272.56	404.39	218.52
MAPE	219.03	102.26	102.85	46.79	89.33	61.26	128.63	37.18

(*) Coefficient is significant at 0.05.

From 219.03Table (8), it can note that the estimated regression parameters of all estimators have the same signs except LDK estimator. Moreover, the MSE and MAPE values of MUOE estimator are lower than other estimators, which means that the MUOE estimator achieves the best performance. The results agree with the simulation results.

Therefore, the regression equation estimated using the MUOE model when K = 0.8315 and d = 0.6038 as follows:

$Y_i = 6.3202 + 0.7395X_1 + 0.8851X_2 + 0.3927X_3 + 0.5033X_4$

Conclusion

In this study, the performance of seven recent estimators to combat multicollinearity in linear regression models was compared. A simulation study has been conducted to compare the performance of the P_1 (k , d), P_2 (k , d), AUMRTE, LDK, AKDE, LKL, and MUOE estimators. It is evident from simulation results that MUOE estimator gives better results than the rest of the estimators considered in this research work at the different sample sizes, sigma, multicollinearity levels, and bias parameters. Finally, application of real-life data further established the superiority of the MUOE estimator as it gives the best result among the existing estimators using the Mean Square Error (MSE) and Mean Absolute Percentage Errors (MAPE) criterions.

References

1. Aslam, M. (2014). Performance of Kibria's method for the hetroscedastic ridge degression model: Some Monte Carlo evidence. Communications in Statistics–Simulation and Computation, 43, (4).
2. Aslam, M. and Ahmad, S. (2020). The modified Liu-ridge-type estimator a new class of biased estimators to address multicollinearity. Communications in Statistics-Simulation and Computation. Doi:10.1080/03610918.1806324.
3. Al-Taweel, Y., Algamal, A., 2022. Some almost unbiased ridge regression estimators for the zero-inflated Poisson model. J. Appl. Eng. Math. 12 (1).
4. Algamal, A., 2021. Almost unbiased ridge estimator in the count data regression models. Electron. J. Appl. Stat. Anal. 14 (1).
5. Alheety, M.I., Qasim, M., Mansson, K., Kibria, B.M.G., 2021. Modified almost unbiased two-parameter estimator for the Poisson regression model with an application to accident data. SORT 45 (2).
6. CROUSE, R. H., JIN, C., HANUMARA, R. C., (1995). Unbiased ridge estimation with prior information and ridge trace. Commun. Statist. Theor. Meth. 24, 2341-2354.
7. Dorugade, A., 2014. A modified two-parameter estimator in linear regression. Stat. Transl. 15 (1).
8. Dawoud, I., Kibria, G., (2020). A new biased estimator to combat the multicollinearity of the Gaussian linear regression model. Stats 3.
9. Hoerl, A. E., Kennard, R.W. (1970). Ridge regression: Biased estimation for non-orthogonal problems. Technometrics 12:55-67.

10. Idowu, J. I., Oladapo, O. J., Owolabi, A.T., Ayinde, K. and Akinmoju, O. (2023), Combating multicollinearity: A new two-parameter approach, Nicel Bilimler Dergisi, 5(1), 90-116. Doi:10.51541/nicel.1084768
11. Jassim, H.A. and Alheety, M.I. (2023a). Modified new estimators depending on unbiased ridge estimator for linear regression model. Journal of Survey in Fisheries Sciences, 10 (3S).
12. Jassim, H.A. and Alheety, M.I. (2023b). Modified unbiased optimal estimator for linear regression model. Journal of University of Anbar for Pure Science (JUAPS), 17 (2).
13. Kibria, B.M.G., Lukman, A.F. (2020) A new ridge-type estimator for the linear regression model: Simulations and applications. Scientifica 2020.
14. Khalaf, G., and Shukur, G. (2005). Choosing ridge parameters for regression problems. Communications in Statistics: Theory and Methods 34.
15. Liu, K., 1993. A new class of biased estimate in linear regression. Commun. Stat., Theory Methods 22 (2).
16. Liu, K., 2003. Using Liu-type estimator to combat collinearity. Commun. Stat., Theory Methods 32 (5).
17. Lukman, A.F., Ayinde, K., Binuomote, S. and Clement, O.A. (2019a). Modified ridge-type estimator to combat multicollinearity: application to chemical data. Journal of Chemometrics, 33(5), e3125.
18. Lukman, A.F., Ayinde, K., Sek, S.K. and Adewuyi, E. (2019b), A modified new two-parameter estimator in a linear regression model, Modelling and Simulation in Engineering 2019:6342702.
19. Makhdoom, W. and Aslam, M. (2023). More efficient estimation strategy for (k-d) class estimator in existence of multicollinearity and heteroscedasticity: some Monte Carlo simulation evidence. Journal of Statistics, 27.
20. Muniz, g. and Kibria, B.M.G. (2009). On some ridge regression estimators: An empirical comparison. Communications in Statistics-Simulation and Computation, 38.
21. Omara, T., 2019. Modifying two-parameter ridge Liu estimator based on ridge estimation. Pak. J. Stat. Oper. Res. 14 (4).

22. Omara, T. (2022). Almost unbiased modified ridge-type estimator: An application to tourism sector data in Egypt. Heliyon 8 (2022)e10684.
23. Owolabi, A. T., Ayinde, K. and Alabi, O.O. (2022a), A new ridge-type estimator for the linear regression model with correlated regressors, Concurrency and Computation: Practice and Experience, p. CPE6933.
24. Owolabi, A. T., Ayinde, K., Idowu, J.I., Oladapo, O.J. and Lukman, A.F. (2022b), A New two-parameter estimator in the linear regression model with correlated regressors, Journal of Statistics Applications & Probability, 11.
25. Ozkale, M.R., Kaciranlar, S., 2007. The restricted and unrestricted two-parameter estimators. Commun. Stat., Theory Methods 36, 2707-2725.
26. Oladapo, O.J., Owolabi, A.T., Idowu, J.I., and Ayinde, K. (2022). A new modified Liu ridge-type estimator for the linear regression model: simulation and application. Int. J Clin Biostat Biom. 8 (2022).
27. Oasim, M., Mansson, K., Sjolandar, P., and Kibria, G. (2022). A new class of efficient and debiased two-step shrinkage estimators: method and application. Journal of Applied Statistics, 49 (16).
28. Swindel, B. F. (1976). Good ridge estimators based on prior information. Communications in Statistics-Theory and Methods A5: 1065-1075.
29. Sakalhoglu, S., Kacranlar, S., 2008. A new biased estimator based on ridge estimation Stat. Pap. 49, 669-689.
30. Xu, W., Yang, H., 2011. More on the bias and variance comparisons of the restricted almost unbiased estimators. Commun. Stat. A40(22).
31. Yang, H., chang, X., 2010. A new two-parameter estimator in linear regression. Commun. Stat., Theory Methods 39 (6).